CULTURALLY-MATHEMATICALLY RELEVANT PEDAGOGY (CMRP): FOSTERING URBAN ENGLISH LANGUAGE LEARNERS' MULTIPLICATIVE REASONING

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In this paper we articulate an approach, termed culturally-mathematically relevant pedagogy (CMRP), for fostering urban English language learners' mathematical progression. CMRP integrates three aspects, the use of (1) adaptive teaching to build on students' funds of knowledge for mathematics, (2) tasks that make sense to students given their current mathematical conceptions, and (3) manipulatives and representations that, for the students, meaningfully signify quantities linked to numbers and operations used in a task. To situate CMRP, we use a continuum of conceptual transitions in multiplicative reasoning, which are critical for supporting students' development of algebraic reasoning.

Keywords: Culturally Relevant Pedagogy; Learning Trajectories; Number Concepts and Operations; Instructional Activities and Practices (Adaptive Teaching)

In this theoretical paper, we argue for the need to expand pedagogical perspectives so they integrate, build on, and are relevant to *both* cultural and mathematical aspects of students' prior experiences and knowledge. Our expansion draws on the core notion of *Culturally Relevant Pedagogy (CRP)* (Ladson-Billings, 1995). This notion has greatly enhanced sensitivity to issues of congruency between student experiences of educative processes at their home/community and in schools. The thrust to augment such congruency for diverse student populations was proposed in place of deficit views of students from underserved populations (e.g., lack of family support), which essentially blame the victims for their poor achievements (e.g., Sleeter, 1997).

For English Language Learners (ELLs), CRP entails addressing the complex interactions between their mathematical communication and understandings, while engaging them in work on cognitively challenging tasks (Moschkovich, 2002). CRP encourages ELLs' use of multiple languages as it enables more complex mathematical activity than when using just English (Moschkovich, 2007). We argue that sensitivity to students' cultures and languages is necessary but insufficient to foster their mathematical progressions. This is supported by research findings suggesting that language fluency does not fully account for differences in ELLs' mathematical proficiency (e.g., Abedi et al., 2006). Our thesis is that to foster students' mathematics learning pedagogy has to be relevant to *both* their cultural and mathematical resources.

To realize mathematics teaching for equity an integrated, *Culturally-Mathematically Relevant Pedagogy (CMRP)* is needed. We suggest that being ***M***athematically relevant entails: (1) Using adaptive teaching to build on students' funds of knowledge *for mathematics*, (2) Using tasks that make sense to students given *their* prior conceptions, and (3) Using manipulatives and representations that, *for the students*, meaningfully signify quantities linked to numbers and operations used in a task. Below, we elaborate on each of these three aspects of CMRP.

Adaptive Teaching

By adaptive teaching we refer to pedagogical methods that are tailored, *every mathematics lesson*, to students' resources—conceptions and experiences they have and bring to a learning situation, termed *funds of knowledge* (Moll et al., 1992). These resources *afford and constrain* advances to rigorous mathematical

understandings expected of each young person. We set out to identify students' resources, relevant for lesson goals, and design instruction (tasks, activities, materials) that methodically builds on these resources as a means to engender learning of the intended mathematics. We emphasize that adaptive teaching is *not just* student-centered, or standard- (content) informed, or activity- (problem) based. Rather, it integrates all of these into what Tzur (2010) has termed the "*Teaching Triad*" (Figure 1), which stresses the need to analyze, begin at, and build on developmental continua in student available conceptions. This adaptive approach, depicted by the Teaching Triad, is rooted in Simon's (1995) idea of a teaching cycle revolving around hypothetical learning trajectories (HLT).

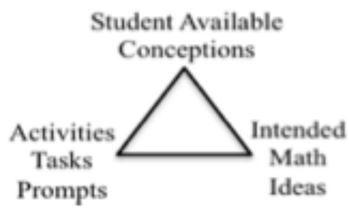

Figure 1: The Teaching Triad

Adaptive teaching, as depicted by the Teaching Triad, includes three principal activities. The first, which ties teaching to students' resources, is the ongoing analysis of students' available mathematical conceptions: goals they might set, activities they might use, contexts familiar to them, objects they might operate on, and effects of their activities that they might notice. The second principal activity is deciphering the intended mathematics into underlying, goal-directed activities. The third principal activity is articulating paths between students' extant thinking and the mathematics they are to learn. The teacher (a) hypothesizes how students' activities and reflections may bring forth the intended learning and (b) designs specific task sequences that can promote the advancement sought (Simon & Tzur, 2004). These three principal activities address five key points proposed by Bransford, Brown, and Cocking (1999): engaging students in tasks, tailoring interaction to students' sense of tasks, motivating students' pursuit of goal, adjusting tasks to students' level of frustration, and possibly modeling execution of a new activity.

Adaptive teaching is consistent with the notion of co-learning (Jaworski, 2001, Tzur, 2004). A teacher selects and uses tasks, guides goal setting, and orients student reflections, while students set their goals, initiate activities, notice effects, and abstract new mathematical relationships. This co-learning approach supports emergence of norms that can increase learning opportunities for ELLs. One critical norm is the constant need to explain various solutions to one another. Another is the expectation to collaboratively solve challenging mathematical tasks, including posing tasks to peers and sharing solution strategies (Boaler, 2006). Such collaboration is coupled with appreciation for diversity and the expectation that every student will participate and share different solutions (errors included!) while not limiting others' participation. Another norm includes awareness that there are many viewpoints, and promote respect for those viewpoints that differ from one's own (Moschkovich & Nelsen-Barber, 2009).

Using Tasks that Make Sense to Students Given *Their* Current Knowledge

CMRP, by engaging every student in solving tasks that are constantly challenging her or his available mathematical thinking, can provide the backbone for equitable lesson/unit design (Simon, 2011). Essential to CMRP is the distinction *between task features and children's thinking*. That is, we contend that the structure of a task as seen by an adult does not, in and of itself, determine the way a child makes sense of and acts to solve it. Rather, implementing a task integrates both—a child's current conceptions and sense making (goal-directed actions, units acted upon), and *task features* designed by adults to promote learning of intended mathematics (Tzur & Lambert, 2011). The need for such a combination is seen in the following example.

Consider the task: Enrique has 3 boxes of cookies. Each box contains 5 cookies. How many cookies does Enrique have in all? Students could solve this task in various ways, such as: (1) using a single cube to

signify a cookie and counting, one-by-one, the number of cubes in 3 towers of 5 cubes each to arrive at 15 cookies, (2) counting by 5's, using each finger to signify a tower of cubes (which signifies a box of cookies), and stopping at 15 because 3 fingers have been accrued, and (3) multiplying 3 by 5 to arrive at (a retrieved fact of) 15 cookies.

In the first method, the child needs to count single items—units of 1, but she can use other items, in this case cubes, to stand for these units. In the second method, the child can consider each group of 5 cookies as a "thing," a composite unit. She can continue to accumulate those 5s, anticipating she would be finished when she has accumulated three such units of 5s. In the third method, the child can use a known operation, in this case multiplication, to determine an amount of cookies. As these different methods suggest, although an adult might see a structure in the task and intend for it to elicit a particular operation, the child solving the task engages in operations rooted in her existing conceptions. When implementing mathematical tasks using CMRP, it is necessary to integrate task features with children's sense of the task (e.g., by allowing children to solve the problem in whichever method and language that suit their current ways of making sense of the task). Whereas in this example task features are designed to accentuate coordination of two types of units, a child's solution (e.g., method 1) may include no such coordination.

Using Manipulatives and Representations that Signify Meaningful Quantities for Students

To be adaptive to students' available conceptions, teaching needs to recognize how, when solving tasks, *students* make sense of and use manipulatives and representations, which can signify different quantities for different students. For example, the student who used method (1) above to determine the amount of cookies seemed to be engaging in a one-by-one operation on figural (semi-concrete) objects. That student was able to use a physical object, in this case a cube, to represent an unseen physical object (a cookie). For the student who used method (2), however, a tower (and a finger) represented a composite unit—a box consisting of 5 cookies.

A key feature of adaptive teaching for promoting abstraction of mathematical ideas is therefore identification of how students currently use and interpret particular manipulatives and representations. In turn, teachers need to strategically and gradually promote students' shift from operating on concrete objects, to operating on figural objects (e.g., finger replaces a tower that replaces a box), to operating on abstract (imagined) objects. Students might begin counting only what is visible to them. Next, students could use a different physical object and then figural objects to represent unseen physical objects (e.g., drawing a small square to stand for a box of cookies). Eventually, students could work in the abstract (e.g., using their number sequence to *simultaneously* count the boxes and the cookies—one is 5, two is 10, and three is 15).

Situating CMRP in a Developmental Continuum of Multiplicative Reasoning

In this section, we illustrate a CMRP approach by focusing on multiplicative reasoning—a mathematical prerequisite for advancing to algebraic reasoning (e.g., Mason, 2008; Smith & Thompson, 2008). Rather than focusing on particular topics for students to acquire (e.g., "multiplication of 2-digit by 1-digit numbers"), we target essential ways of reasoning that can underlie students' development of more advanced understandings.

Composite Unit: A Key Distinction between Multiplicative and Additive Reasoning

A key construct we use to distinguish multiplicative from additive reasoning is *number as a composite unit* (Steffe, 1992). When number is conceived of as composite unit, children can anticipate *decomposing* units into "nested" sub-units. For example, a child can decompose 7 into 5+2 because, *for her*, 5 and 2 are "nested" within 7. Key to additive reasoning is that the referent unit is *preserved* (Schwartz, 1991): 11 *cookies* – 7 *cookies* = 4 *cookies*. In contrast, multiplicative reasoning requires a conceptual transformation—a coordination of operations *on* composite units (Behr, 1994). Consider placing 3 cookies into each of 4 boxes; 3 is one composite unit (*cookies per box*) and 4 is another (*boxes*).

Multiplicative reasoning entails distributing one unit over items of another (e.g., 3 cookies into each box) and finding the total via a *coordinated (double) count* (Steffe, 1992): 1 (box) is *1-2-3* (cookies), 2

(boxes) are *4-5-6* (cookies), and so on. C*oordinated* counting involves keeping track of the composite units while accruing the total of 1s based on the size of the distributed composite unit (3 cookies-per-box). As this example indicates, in multiplicative reasoning the referent unit is *transformed* (Schwartz, 1991), and the product has to be conceptualized as a unit of units of units (Steffe & Cobb, 1998): here, "6 cookies" is a unit composed of 3 units (boxes) of 2 units (cookies-per-box). The *simultaneous count* of two composite units and the resulting *unit transformation* constitute the conceptual advance from additive reasoning. Children have difficulty developing multiplicative reasoning, and impoverished conception of number as a composite unit might contribute to that difficulty (Tzur et al., 2010). For example, a student like the one who used method (1) above seemed unable to conceive of a composite unit as a "thing" and thus must have counted individual items to determine the product.

Scheme-and-Task Continuum to Support Students' Multiplicative Reasoning

To foster students' progression in multiplicative reasoning, our CMRP approach follows a sequence of six schemes—goal-directed ways of acting and reasoning (von Glasersfeld, 1995). Below, examples follow each scheme to illustrate tasks on which students work during instruction, to help them advance from prior scheme to the next. A task corresponds to but is not identical with a scheme. For fostering each scheme, we begin by engaging students in operating on tangible objects (e.g., putting cubes together to make a tower). We then proceed to tasks in which students produce the composite units and cover them before figuring out the total. This leads to students' use of figural (substitute) items. Initially these figural items may be physical objects such as fingers. Subsequently students may draw schematic diagrams of the objects. At first, students may draw all single items (1s) in a composite unit. For example, students may draw each of 3 cookies in 5 boxes. Next, we promote a shift to the drawing of only the composite units. For example, students may draw only the 5 boxes and write the number 3 in each box to represent the number of cookies in each box. Finally, we support students' use of numbers to replace the individual items. For example, students may record the numbers 3, 6, 9, 12, 15 to stand for the 5 boxes each containing 3 cookies. To make tasks relevant to children, *we ask them* about and select units and contexts that fit with *their daily experiences* (e.g., shirts and buttons, families and family members, pets and legs, packages and food items, etc.).

For each task we also select numbers that allow gradual progress from "easy" ones (e.g., composite units of 2, 5, or 10 items) to intermediate (units of 3 or 4) and larger/difficult ones (6, 7, 8, and 9, and beyond). The point is quite simple: students are better positioned to construct a new scheme by operating on composite units with which they are familiar. Once the new scheme is evolving, changing to more challenging numbers fosters repeated use and recognition of the invariant nature of that new way of reasoning across any similar situation.

We initiate transition from additive to multiplicative reasoning by fostering students' construction of a ***Multiplicative Double Counting (mDC)*** scheme—the simultaneous counting of composite units and 1s described above. Once students construct mDC, they can anticipate that a total of 1s (say, 24 buttons) is itself a composite unit made of another composite unit (4 shirts), each a composite unit itself (6 buttons). They may solve tasks such as "Enrique has 3 boxes of cookies. Each box contains 5 cookies. How many cookies does Enrique have in all?" Tasks supporting ***mDC*** require students to determine a total number of 1s such that the amount is a composite unit made up of groups of composite units. A solution entails not only figuring out the number of 1s but also and most importantly justifying why this *must* be the total. Typically, we engage students in working on such tasks in pairs, and their justifications often serve to determine which of two different answers is the correct one (including checking their answers by counting the tangible objects).

Next, students can construct a ***Same-Unit Coordination (SUC)*** scheme, in which they learn to apply their additive operations to sets (units) of composite units. They may solve tasks such as, "You had 5 vases, each containing 10 flowers. Now you have 9 vases, each containing 10 flowers. How many *vases* did you gain?" Tasks supporting ***SUC*** require students to reason about a *set* of composite units as made of sub-units, each of which is a set of composite units, without losing sight of their being composed of a particular numerosity. Initially, they may respond to such tasks by mistakenly attempting to count, or

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calculate, the total of 1s in the set. Gradually, with teaching emphasis on the unit being asked about in a task, they shift their focus to operations on the composite units. Such operations can, for example, support students' learning of adding/subtracting units of 10 (e.g., 90–50 means subtracting five 10s from nine 10s, hence four 10s which are forty 1s).

Next, we promote students' construction of a ***Unit Differentiation and Selection (UDS)*** scheme (McClintock et al., 2011), in which they further separate operations on 1s and composite units. They may solve tasks such as, "You have 7 boxes, each containing 5 pencils. I have 4 boxes, each containing 5 pencils. How are our collections similar? Different? How many more pencils do you have?" Tasks supporting ***UDS*** foster students to conceive of two sets of composite units in terms of sub-units that constitute each, and operate multiplicatively *on the difference* between the two sets. Such operations can, for example, support students' learning of the distributive property of multiplication over addition (e.g., to solve the above problem they can either find the totals first, 35 and 20, and then subtract to obtain 15, or find the difference of 7-4 first, and then multiply it by 5 to obtain 15).

Then, we promote students construction of a ***Mixed-Unit Coordination (MUC)*** scheme (Tzur et al., 2009), in which they operate on two collections—one consisting of composite units and the other of 1s. They may solve tasks such as, "You have 7 boxes, each containing 3 candies. I'll give you 6 more *candies*. If you put these 6 candies into boxes containing 3 each, how many *boxes* will you have in all?" Tasks supporting ***MUC*** foster students' selecting the distributed composite unit (e.g., 3 cookies-per-box), imposing that unit on the singletons (e.g., 6 candies) to yield 2 composite units (e.g., boxes), and then adding these to the initial set (e.g., 7+2=9 boxes). Such operations can, for example, support students' learning of algebra-like mathematics that involves both additive and multiplicative operations on different-but-related units (e.g., 3*10+15=4*10+1*5=45, and later solving for the number 'x' that will make the equation 3x+15=45 a true statement).

From ***MUC*** we proceed to fostering students' construction of a ***Quotitive Division (QD)*** scheme as an inverse operation to multiplicative double counting. They double count to solve tasks like: "I counted 60 legs of chairs in the class. Each chair has 4 legs. How many chairs are in the class?" Tasks supporting ***QD*** require students to operate on a total of 1s and use the size of each composite unit to determine (keep track of) how many such units can be made. Such operations can, for example, support students' learning to use a single structure (equation) to represent, and solve, multiplicative situations by identifying which two of three quantities are given, which needs to be figured out, and what operation is required (e.g., 4x?=60 leads to dividing 60÷4, whereas 4x15=? leads to multiplying the two numbers).

Later, we move on to fostering students' construction of a ***Partitive Division (PD)*** scheme, in which they operate on a total of 1s by distributing it equally into a given number of composite units. They solve tasks such as "Our class has 24 students. We want to place them into 3 groups. How many students will be in each group?" Tasks supporting ***PD*** require students to figure out the size of each distributed composite unit, given the amount of composite units and the total of 1s. Such operations can, for example, support students' learning to meaningfully use the long-division algorithm, as a process in which units of units (etc.) of 10s are distributed into the given number of groups (divisor), while exchanges to smaller units enable such distribution when there are not enough larger units (e.g., to divide 294 into 7 equal groups one exchanges two 100s into twenty 10s, adds the nine 10s, and distribute those twenty-nine 10s by placing four such units in each of the groups, leaving one 10 that needs to be further exchanged, etc.).

Enacting CMRP with Urban ELLs

We have begun to enact a CMRP approach through piloting instruction designed to promote the aforementioned conceptual transitions in multiplicative reasoning in urban ELLs. This pilot work was conducted at an elementary school with over 85% of students whose native language is Spanish. It included work with individual 4th graders who were identified as having disabilities in mathematics, and with K–5 teachers in the school. The latter included workshops with teachers that focused on their own mathematical understandings (e.g., of the place-value, base-ten number system), on children's developmental pathways to mathematical conceptions (from rote-counting, through cardinality and counting-all to counting-on, and to additive and multiplicative reasoning with whole numbers), and on the

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CMRP, adaptive teaching approach. Besides the workshops, we have been partnering with two teachers (grade 3 and grade 4) to co-plan and co-teach their students as a medium for enacting and demonstrating the adaptive teaching.

Our pilot work highlights how a CMRP may provide ELLs with instruction that gives them the opportunity to work on cognitively challenging mathematical tasks (e.g., Brown et al., 2011; Campbell, Adams, & Davis, 2007; Moschkovich, 2007). Moschkovich (2002) asserted that equitable mathematics instruction for ELLs must move beyond focusing on acquiring mathematical vocabulary and recognizing multiple meanings for terms. By analyzing students' available conceptions and fostering their problem solving in conceptually tailored tasks, CMRP seems to enable engaging in and linking mathematical processes and expressions. In particular, by fostering students' movement from concrete to figurative to abstract representations, CMRP seemed to promote a focus on *their* mathematical reasoning and justifying. When students justified their solutions, they had an opportunity to do so in multiple ways—in English and in one's native language. Thus, students were highly engaged in discussions with their peers as they shared their strategies. Further, when justifying their results, students have used physical objects and/or diagrams to represent the composite units (e.g., boxes of cookies) involved in the problem. Strategic use of representations supported students' development of more advanced mathematical structures and fostered their productive participation in mathematical practices. Preliminary results of enacting CMRP to promote ELLs' advancement from additive to multiplicative reasoning indicated substantial impact.

Discussion

In this paper we proposed a Culturally-Mathematically Relevant Pedagogy (CMRP) approach, including three key aspects: adaptive teaching, sensible tasks, and meaningful manipulatives. Central to all three aspects is students' engagement in mathematical practices they can make sense of via their prior conceptions. The three aspects of CMRP are interrelated, as indicated by the Teaching Triad. That is, a CMRP approach links instructional goals to funds of knowledge that low SES and ELLs bring to a lesson. If traditional lessons focused on one vertex of the *Teaching Triad* (mathematics), reform lessons on two (tasks/activities and mathematics), CMRP lessons are innovative in that the focus is on all three vertices (student resources/conceptions, tasks/activities, and intended mathematics). Each CMRP lesson begins with tasks that build on student mathematics. This supports gradual use of challenging tasks by making sure the tasks are tailored to students for whom the instruction is being designed.

In essence, CMRP is a mathematically focused implementation of Culturally Relevant Pedagogy (CRP). When implementing challenging tasks, CMRP uses situations that are supposedly meaningful to the students. These tasks build on students' abilities rather than deficiencies and allow for multiple pathways to their success. The "M" part of this approach focuses on strategic tailoring of problem situations and representations that make sense to students given their current understandings. The goal of this strategic tailoring is to promote students' mathematical progression, in our case to promote transition along our six-scheme continuum of multiplicative reasoning. For example, if a student has developed multiplicative double counting (mDC) only for visible items, the student would engage in tasks that would support her development of mDC for *hidden* items. Another student who had developed mDC for abstract items could engage in tasks that foster Same Unit Coordination (SUC) with visible composite units. The articulation of the schemes involved in multiplicative reasoning augments our sensitivity to students' mathematical progression. By linking the operation of division with the operation of multiplication, students can meaningfully solve problems that they otherwise may have been able to solve only with the use of an algorithm. In turn, we foster bringing forth each of the schemes to support its use for more advanced algebraic reasoning.

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